Postcritical sets and saddle basic sets for Axiom A polynomial skew products on $\mathbb{C}^2$

Shizuo Nakane Tokyo Polytechnic University

Investigating the link between postcritical behaviors and the relations of saddle basic sets for Axiom A polynomial skew products on $\mathbb{C}^2$, we characterize various properties concerning the three kinds of accumulation sets defined in [DH1] in terms of the saddle basic sets. We also give a new example of higher degree.

1 Introduction

In holomorphic dynamics, the behaviors of the orbits of critical points play a central role. And so do the relations between the basic sets in the dynamics of Axiom A maps. In this article, we investigate the link between postcritical behaviors and the relations of saddle basic sets for Axiom A polynomial skew products on $\mathbb{C}^2$. Through this investigation, we improve some of the results in [DH1, DH2] and give complete characterizations of equalities between three kinds of accumulation sets of the critical set. As a corollary, we give stability results of these equalities. We also give a new example of higher degree.

A polynomial skew product on $\mathbb{C}^2$ is a map of the form : $f(z,w) = (p(z), q(z,w))$, where $p(z)$ and $q_z(w) := q(z,w)$ are polynomials of degree $d \geq 2$. It is called *regular* if it extends to a holomorphic map on $\mathbb{P}^2$. Its k-th iterate is expressed by :

$$f^k(z,w) = (p^k(z), q_{p^{k-1}(z)} \circ \cdots \circ q_z(w)) =: (p^k(z), Q_z^k(w)).$$

Thus it preserves the family of *fibers* $\{z\} \times \mathbb{C}$ and this makes it possible to study its dynamics more precisely. Let K be the set of points with bounded orbits and set $K_z = \{w \in \mathbb{C}; (z,w) \in K\}$. The *fiber Julia set* J_z is the boundary of K_z. More generally, for $T \subset \mathbb{C}^2$ and $Z \subset \mathbb{C}$, we set $T_z = \{w \in \mathbb{C}; (z,w) \in T\}$ and $T_Z = \cup_{z \in Z}\{z\} \times T_z$.

Let K_p and J_p be the filled-in Julia set and Julia set of p respectively. We are interested in the dynamics of f on $J_p \times \mathbb{C}$ because the dynamics outside $J_p \times \mathbb{C}$ is fairly simple. Consider the *critical set*

$$C_Z = \{(z,w) \in Z \times \mathbb{C}; q_z'(w) = 0\}$$

over $Z \subset \mathbb{C}$. We will investigate the behaviors of the orbits of points in C_{J_p}. For any subset X in $\mathbb{C}^2$, its accumulation set is defined by

$$A(X) = \cap_{N \geq 0} \overline{\cup_{n \geq N} f^n(X)}.$$

DeMarco & Hruska [DH1] defined the *pointwise* and *component-wise* accumulation sets of C_{J_p} respectively by

$$A_{pt}(C_{J_p}) = \overline{\cup_{x \in C_{J_p}} A(x)} \ \text{ and } \ A_{cc}(C_{J_p}) = \overline{\cup_{C \in \mathcal{C}(C_{J_p})} A(C)},$$

where $\mathcal{C}(C_{J_p})$ denotes the collection of connected components of C_{J_p}. It follows from the definition that

$$A_{pt}(C_{J_p}) \subset A_{cc}(C_{J_p}) \subset A(C_{J_p}).$$

It also follows that $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p})$ if J_p is a Cantor set while $A_{cc}(C_{J_p}) = A(C_{J_p})$ if J_p is connected.

Let Λ be the closure of the set of saddle periodic points in $J_p \times \mathbb{C}$. The *stable* and *unstable sets* of Λ are respectively defined by

$$\begin{aligned} W^s(\Lambda) &= \{y \in \mathbb{C}^2; f^k(y) \to \Lambda\}, \\ W^u(\Lambda) &= \{y \in \mathbb{C}^2; \exists \text{ prehistory } \hat{y} = (y_k)_{k \le 0} \text{ of } y \text{ tending to } \Lambda\}. \end{aligned}$$

Throughout this section, we assume that f is an Axiom A regular polynomial skew product on $\mathbb{C}^2$. In [DH1], they characterized $A_{pt}(C_{J_p})$ and $A(C_{J_p})$.

Theorem A. ([DH1], Theorem 1.1)

$$A_{pt}(C_{J_p}) = \Lambda, \quad A(C_{J_p}) = W^u(\Lambda) \cap (J_p \times \mathbb{C}).$$

Thus, any $x \in C_{J_p}$ either tends to Λ or escapes to ∞. They also tried to characterize the equalities between any two of the accumulation sets and give examples with various properties.

We will improve their works and give a new example of higher degree. A key tool is the *saddle basic set*, i.e., the basic set of unstable dimension one. Here a *basic set* is a compact invariant subset of the non-wandering set Ω of f with dense orbit. The saddle set Λ decomposes into a disjoint union of saddle basic sets : $\Lambda = \sqcup_{i=1}^m \Lambda_i$, which is the saddle part of Ω in $J_p \times \mathbb{C}$. For convenience, we add one more "basic set," which corresponds to the superattracting fixed point $\{[0:1:0]\}$ in $\mathbb{P}^2$:

$$\Lambda_0 = \emptyset, \ W^s(\Lambda_0) = (J_p \times \mathbb{C}) \setminus K, \ W^u(\Lambda_0) = \emptyset.$$

Note that this is not a saddle basic set. We also set

$$C_i = C_{J_p} \cap W^s(\Lambda_i) \ (0 \le i \le m).$$

First we characterize the equality $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$.

Theorem 1.1.

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \Longleftrightarrow \forall C \in \mathcal{C}(C_{J_p}),\ 0 \leq \exists i \leq m \ \ such\ that\ \ C \subset C_i. \quad (1)$$

This is an improvement of the following.

Theorem B. ([DH1, DH2])

$$A_{cc}(C_{J_p}) = A_{pt}(C_{J_p}) \Longrightarrow \forall C \in \mathcal{C}(C_{J_p}),\ C \cap K = \emptyset \ or\ C \subset K. \quad (2)$$

In fact, in terms of C_i, the condition in (2) is expressed by

$$\forall C \in \mathcal{C}(C_{J_p}), \quad C \subset C_0 \text{ or } C \subset \cup_{i=1}^m C_i.$$

Hence, the condition in (1) coincides with that in (2) only if $m = 1$, that is, Λ itself is a basic set.

The following two theorems give characterizations of $A(C_{J_p}) = A_{pt}(C_{J_p})$ in terms of C_i.

Theorem 1.2. *For each $i \geq 0$,*

$$A(C_i) = \Lambda_i \Longleftrightarrow C_i \ is\ closed. \quad (3)$$

Consequently,

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \Longleftrightarrow C_i \ is\ closed\ for\ any\ i \geq 0.$$

Theorem 1.3. *For each $j \geq 1$,*

$$\begin{aligned} C_j \ is\ open\ in\ C_{J_p} &\Longleftrightarrow W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j \\ &\Longleftrightarrow z \mapsto \Lambda_{j,z} \ is\ continuous\ in\ J_p. \end{aligned}$$

Consequently,

$$\begin{aligned} \forall j \geq 1, C_j \ is\ open\ in\ C_{J_p} &\Longleftrightarrow W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda \\ &\Longleftrightarrow z \mapsto \Lambda_z \ is\ continuous\ in\ J_p. \end{aligned}$$

Note that $C_0 = C_{J_p} \setminus K$ is always open in C_{J_p}. Here the continuity is with respect to the Hausdorff topology. These can be regarded as a refinement of the following.

Theorem C. ([DH2], Theorem E5.2)

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \iff \text{ the map } z \mapsto \Lambda_z \text{ is continuous in } J_p. \tag{4}$$

Under the assumption $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda$,

$$A(C_{J_p}) = A_{pt}(C_{J_p}) \iff \text{ the map } z \mapsto K_z \text{ is continuous in } J_p. \tag{5}$$

Here we give some examples.

Example 1.1. The product map $f(z,w) = (p(z), q(w))$ is Axiom A if and only if both p and q are hyperbolic. Let $A_i = \{w_{i,1}, \cdots, w_{i,k_i}\}, 1 \le i \le m$ be the attracting cycles of q and B_i be the set of critical points of q contained in the basin of A_i. Then $\Lambda_i = J_p \times A_i$ are the saddle basic sets and $C_i = J_p \times B_i$ are open and closed in C_{J_p}.

Example 1.2. Sumi [Su] gives an example of a polynomial skew product :

$$f(z,w) = \left(p(z), w^{2^n} + \frac{z + \sqrt{R}}{2\sqrt{R}} t_{n,\epsilon}(w) \right),$$

where $R, \epsilon > 0, n \in \mathbb{N}, p = p_R^n, p_R(z) = z^2 - R, t_{n,\epsilon}(w) = h_\epsilon^n(w) - w^{2^n}, h_\epsilon(w) = (w - \epsilon)^2 - 1 + \epsilon$. If R is large, ϵ is small and n is even and large, then f is Axiom A and J_p is a Cantor set. Let $\alpha < 0$ and $\beta > 0$ be the fixed points of p_R. Then it satisfies

(a) $\Lambda = \Lambda_1 \sqcup \Lambda_2$, where Λ_1 consists of a single point in $\{\beta\} \times \mathbb{C}$,
(b) $C_{J_p} \subset K$ i.e. $C_0 = \emptyset$, hence $z \mapsto K_z$ is continuous in J_p,
(c) C_1 is a finite set in $\{\beta\} \times \mathbb{C}$,
(d) $C_2 = C_{J_p} \setminus C_1$ is open in C_{J_p} and $\overline{C_2} \supset C_1$,
(e) $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) \neq A(C_{J_p})$.

By Theorems 1.2 and 1.3, it follows that $A(C_1) = \Lambda_1$ and $W^u(\Lambda_2) \cap (J_p \times \mathbb{C}) = \Lambda_2$. From (d), we have $W^u(\Lambda_1) \cap W^s(\Lambda_2) \neq \emptyset$ (see Proposition 3.1 below).

Thus, the equivalence in (5) does not hold in general. So far, Example 1.2 is the only example of an Axiom A map which has two saddle basic sets with a relation. This suggests that we need to take into account the relations among saddle basic sets. It turns out that the equality $A(C_{J_p}) = A_{pt}(C_{J_p})$ or $W^u(\Lambda) \cap (J_p \times \mathbb{C}) = \Lambda$ decomposes into two independent equalities :

$$W^u(\Lambda) \cap (J_p \times \mathbb{C}) = W^u(\Lambda) \cap W^s(\Lambda) \text{ and } W^u(\Lambda) \cap W^s(\Lambda) = \Lambda,$$

both of which are characterized in terms of C_i. See Theorems 3.1 and 3.2. As will be seen in Lemma 2.3,

$$W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \iff W^u(\Lambda_i) \cap W^s(\Lambda_j) = \emptyset \text{ for any } 1 \le i \neq j \le m. \tag{6}$$

As for the stability of the equalities $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ and $A(C_{J_p}) = A_{pt}(C_{J_p})$, we have the following.

Theorem 1.4. *Both equalities* $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ *and* $A(C_{J_p}) = A_{pt}(C_{J_p})$ *are preserved in hyperbolic components.*

The idea of proof of Theorem 1.4 is due to that of Proposition 6.3 in [DH1]. Note that [DH2] has already given another proof for the equality $A(C_{J_p}) = A_{pt}(C_{J_p})$, based on the characterization in Theorem C. By virtue of Theorems 1.1 and 1.2, we can prove both cases in the same way.

The following two theorems give answers to some questions in [DH1].

Theorem 1.5. *Suppose* J_p *is disconnected. Then*

$$A_{cc}(C_{J_p}) = A(C_{J_p}) \Longleftrightarrow A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) = A(C_{J_p}).$$

Note that $A_{cc}(C_{J_p}) = A(C_{J_p})$ if J_p is connected. Thus this gives a characterization of the equality $A_{cc}(C_{J_p}) = A(C_{J_p})$. Together with Theorem 1.4, it follows that the equality $A_{cc}(C_{J_p}) = A(C_{J_p})$ is preserved in hyperbolic components. This answers Question 8.2 in [DH1].

Theorem 1.6. *There exists an Axiom A polynomial skew product* f *of degree four with the following properties :*
(a) J_p *is neither connected nor totally disconnected,*
(b) $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) \neq A(C_{J_p})$.

In [DH1], they give examples satisfying each of the following properties respectively except (vi):

(i) $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) = A(C_{J_p})$,
(ii) $A_{pt}(C_{J_p}) \neq A_{cc}(C_{J_p}) = A(C_{J_p})$,
(iii) $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) \neq A(C_{J_p})$,
(vi) $A_{pt}(C_{J_p}) \neq A_{cc}(C_{J_p}) \neq A(C_{J_p})$.

But all of their base Julia sets J_p are either connected or totally disconnected and they posed a question as Question 8.1 on the existence of examples whose base Julia sets are neither connected nor totally disconnected. Theorem 1.5 says that there is no such example satisfying (ii), while Theorem 1.6 gives one satisfying (iii). It is still unknown whether there exists an example satisfying (vi).

The author would like to thank Hiroki Sumi for helpful discussion on his example and Laura DeMarco for valuable advice. He also thanks the referees for helpful suggestions that refine this article.

2 Preliminaries

In this section, first we prepare several notions and properties in the theory of dynamics of hyperbolic C^∞ endomorphisms. This theory was established in Przytycki [P] and Ruelle [R]. Here we collect them from Jonsson [J1]. Let f be a C^∞ endomorphism of a C^∞ Riemannian manifold M. Consider a compact set $L \subset M$ which satisfies $f(L) = L$. The hyperbolicity of L for a non-invertible map f is defined through the *natural extension* :

$$\hat{L} = \{\hat{x} = (x_k)_{k\leq 0}; x_j \in L, f(x_k) = x_{k+1} \ (k \leq -1)\}.$$

and the invertible shift map $\hat{f} : \hat{L} \to \hat{L},\ \hat{f}((x_k)) = (x_{k+1})$.

An endomorphism f is said to be *Axiom A* if the *non-wandering set* Ω is compact, periodic points are dense in Ω (hence $f(\Omega) = \Omega$) and Ω is hyperbolic. The following lemma is a consequence of the fact that the natural extension of a hyperbolic set for an open Axiom A endomorphism has local product structure. See Proposition 3.3 in [J1].

Lemma 2.1. ([J1], Corollary 2.6)
Let L be a hyperbolic set for an open Axiom A endomorphism f. Then, for any sufficiently small neighborhood U of L, we have

(a) if $y \in U$ and $f^k(y) \in U$ for any $k \geq 0$, then $y \in W^s_{loc}(x)$ for some $x \in L$,
(b) if $y \in U$ has a prehistory $\hat{y} = (y_k)$ with $y_k \in U$ for any $k \leq 0$, then $y \in W^u_{loc}(\hat{x})$ for some $\hat{x} \in \hat{L}$.

By Corollary 3.5 in [J1] or Theorem A.3 in [J3], for an open Axiom A endomorphism f, the non-wandering set Ω has a *spectral decomposition* $\Omega = \sqcup_i \Omega_i$ into basic sets. Here a subset Ω_i of Ω is called a *basic set* if it is compact, satisfies $f(\Omega_i) = \Omega_i$ and f is transitive on Ω_i. A *relation* $\succ$ is defined between basic sets by $\Omega_i \succ \Omega_j$ if $(W^u(\Omega_i) \setminus \Omega_i) \cap (W^s(\Omega_j) \setminus \Omega_j) \neq \emptyset$. A *cycle* is a chain of basic sets satisfying

$$\Omega_{i_1} \succ \Omega_{i_2} \succ \cdots \succ \Omega_{i_n} = \Omega_{i_1}.$$

There is no trivial cycle for open Axiom A endomorphisms :

Lemma 2.2. ([J1], Lemma 4.1 or [J3], Proposition A.4)
For open Axiom A endomorphisms, $W^u(\Omega_i) \cap W^s(\Omega_i) = \Omega_i$ holds for any i.

In case $i \neq j$, the property $\Omega_i \succ \Omega_j$ is somewhat simplified. The following also shows the equivalence in (6).

Lemma 2.3. *For $i \neq j$, $\Omega_i \succ \Omega_j$ if and only if $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \emptyset$.*

proof. We have only to show the 'if' part. Suppose $W^u(\Omega_i) \cap W^s(\Omega_j) \neq \emptyset$ but $(W^u(\Omega_i) \setminus \Omega_i) \cap (W^s(\Omega_j) \setminus \Omega_j) = \emptyset$. Since $W^u(\Omega_i) \cap W^s(\Omega_j)$ never intersects Ω_i, it is included in Ω_j. Any point $x \in W^u(\Omega_i) \cap W^s(\Omega_j)$ has a prehistory $\hat{x} = (x_k)$ tending to Ω_i. Then, for all $k \leq 0$, we have $x_k \in W^u(\Omega_i) \cap W^s(\Omega_j)$, hence $x_k \in \Omega_j$. This is a contradiction. $\square$

Now we restrict our maps to regular polynomial skew products $f(z,w) = (p(z), q(z,w))$ on $\mathbb{C}^2$ and give some of their basic properties from [J3], which will be repeatedly used later. See also Sester [Se] for hyperbolicity of fibered polynomials.

Let Z be a closed subset of $\mathbb{C}$ such that $p(Z) \subset Z$. Actually, Z is either J_p or the set A_p of attracting periodic points of p. Let $D_Z = \overline{\cup_{n \geq 1} f^n(C_Z)}$ be the *postcritical set* of C_Z and set $J_Z = \overline{\cup_{z \in Z} \{z\} \times J_z}$. Jonsson [J3] gave a characterization for f to be Axiom A.

Theorem 2.1. ([J3], Theorems 8.2 and 3.1) *A regular polynomial skew product f on $\mathbb{C}^2$ is Axiom A if and only if the following three conditions are satisfied :*

(a) p is hyperbolic,
(b) $D_{J_p} \cap J_{J_p} = \emptyset$,
(c) $D_{A_p} \cap J_{A_p} = \emptyset$.

Theorem 2.2. ([J3], Proposition 3.5)
If $D_Z \cap J_Z = \emptyset$, then the map $z \mapsto J_z$ is continuous in Z.

Let μ be the ergodic measure of maximal entropy for f (see [FS1] or [J3]). Its support J_2 is called the *second Julia set* of f. Corollary 4.4 in [J3] says that $J_2 = J_{J_p}$. By Theorems 2.1 and 2.2, $J_2 = \cup_{z \in J_p} \{z\} \times J_z$ if f is Axiom A.

The following is a key lemma for the proof of Theorem 1.1.

Theorem 2.3. ([J3], Theorem 8.2 and Proposition A.7)
Axiom A regular polynomial skew products on $\mathbb{C}^2$ have no cycles.

Note that the local stable manifold $W^s_{loc}(x)$ of $x \in \Lambda$ is included in the fiber containing x because f is contracting in the fiber direction on Λ. Hence the local unstable manifold $W^u_{loc}(\hat{x})$ is transversal to the fiber for any $\hat{x} \in \hat{\Lambda}$.

We remark that Fornaess & Sibony [FS2] also investigated hyperbolic holomorphic maps on $\mathbb{P}^2$. See also Mihailescu [Mih] and Mihailescu & Urbanski [MU].

3 Proofs of Theorems

3.1 Proof of Theorem 1.1

Note that $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ if and only if $A(C) \subset \Lambda$ for any $C \in \mathcal{C}(C_{J_p})$. First we show the following.

Lemma 3.1. *For Axiom A regular polynomial skew product f on $\mathbb{C}^2$, we have $C_{J_p} = \sqcup_{i=0}^m C_i$.*

proof. Theorem A implies $A(x) \subset \Lambda$ for any $x \in C_{J_p}$. If $A(x) = \emptyset$, then $x \in C_0$. Otherwise, by Lemma 2.1, there exist an n and $y \in \Lambda$ such that $f^n(x) \in W^s_{loc}(y)$. Hence $A(x) \subset \Lambda_i$ if $y \in \Lambda_i$. Thus we have $C_{J_p} = \sqcup_{i=0}^m C_i$. $\Box$

Now we prove Theorem 1.1.

($\Rightarrow$) Suppose $C \in \mathcal{C}(C_{J_p})$ intersects at least two of C_i. By Theorem B, we may assume $C \subset \cup_{i=1}^m C_i$. Then, by Lemma 3.1, we have $C = \sqcup_{i=1}^m (C \cap C_i)$. Since C is closed, if all $C \cap C_i$ are closed, it contradicts the connectivity of C. Thus at least one of them is not closed. We may assume $C \cap \overline{C_i} \cap C_j \neq \emptyset$ for some $i \neq j$. The following holds for $i, j \geq 0$ and will also be used later.

Lemma 3.2. *Suppose $i \neq j$ and $\overline{C_i} \cap C_j \neq \emptyset$. Then $A(C_i) \cap (W^u(\Lambda_j) \setminus \Lambda) \neq \emptyset$.*

proof. Since $\overline{C_i} \cap C_0 = \emptyset$ for $i \neq 0$, we may assume $j \geq 1$. Take a sequence $x_n \in C_i$ tending to $x_0 \in C_j$. Take a small open neighborhood U_k of Λ_k for $1 \leq k \leq m$ so that $f(U_k) \cap U_\ell = U_k \cap U_\ell = \emptyset$ for $k \neq \ell$. Since $x_0 \in C_j$, there exists a $K \geq 0$ such that $f^k(x_0) \in U_j$ for $k \geq K$. Then $f^K(x_n) \in U_j$ for large n. Since $x_n \in C_i$, the orbit of x_n eventually leaves U_j. Hence if we set $k_n = min\{k \geq K; f^k(x_n) \notin U_j\}$, then $k_n \to \infty$. Let y be an accumulation point of the sequence $\{f^{k_n}(x_n)\}$. Then $y \in \overline{f(U_j)} \setminus U_j$ since $f^{k_n - 1}(x_n) \in U_j$. Consequently we have $y \notin \cup U_k$, hence $y \in A(C_i) \setminus \Lambda$.

Next we show $y \in W^u(\Lambda_j)$. Taking subsequences if necessary, set $y_\ell = \lim_{n \to \infty} f^{k_n + \ell}(x_n)$ for $\ell \leq 0$. Then $(y_\ell)_{\ell \leq 0}$ is a prehistory of y with $y_\ell \in \overline{U_j}$ for $\ell \leq -1$. By Lemma 2.1, $y_{-1} \in W^u_{loc}(\hat{x})$ for some $\hat{x} \in \hat{\Lambda}_j$, hence $y \in W^u(\Lambda_j)$. $\Box$

In the above proof, if we take $x_n \in C \cap C_i$, then $x_0 \in C \cap C_j$ and $y \in A(C)$. Thus $A(C)$ contains a point y outside $\Lambda = A_{pt}(C_{J_p})$. Now we conclude $A_{cc}(C_{J_p}) \neq A_{pt}(C_{J_p})$.

($\Leftarrow$) We have only to show that $A(C) \subset \Lambda_i$ if $C \in \mathcal{C}(C_{J_p})$ satisfies $C \subset C_i$. More generally, we show the following.

Lemma 3.3. *If $C \subset C_i$ is closed, then $A(C) \subset \Lambda_i$.*

proof. If $C \subset C_0$, then $A(C) = \emptyset$ since C is compact. Suppose $C \subset C_i$ and there exists $x \in A(C) \setminus \Lambda_i$ for $i \geq 1$. By taking U_i small, there exist $m_n \nearrow \infty$ and $x_n \in C$ such that $f^{m_n}(x_n) \notin U_i$ for any n. Since C is closed, we may assume x_n tends to some $x_0 \in C \subset C_i$. Set $k_n = min\{k \geq K; f^k(x_n) \notin U_i\}$ as above and take an accumulation point y of $\{f^{k_n}(x_n)\}$. By the above argument, $y \in W^u(\Lambda_i) \setminus \Lambda_i$, hence $y \notin W^s(\Lambda_i)$ because of Lemma 2.2. Since $y \in A(C)$, it follows that $y \in K_{J_p} \setminus J_2$, which is, by Lemma 3.6 in [DH1], equal to $W^s(\Lambda)$. Thus y must belong to $W^s(\Lambda_{i_1})$ for some $i_1 \neq i$. That is, we have a sequence $\{f^{k_n}(x_n)\}$ in $W^s(\Lambda_i)$ tending to $y \in W^u(\Lambda_i) \cap W^s(\Lambda_{i_1})$, hence $\Lambda_i \succ \Lambda_{i_1}$ by Lemma 2.3.

By successively applying this argument, we have a chain of saddle basic sets :

$$\Lambda_i \succ \Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots, \quad i \neq i_1 \neq i_2 \neq \cdots.$$

Since there exist only finitely many basic sets, we must have a cycle of them, which contradicts Theorem 2.3. This completes the proof. □

3.2 Proofs of Theorems 1.2 and 1.3

Note that Theorem 1.2 follows from Lemmas 3.2 and 3.3. Here we use the following proposition, which completely characterizes the existence of relations between saddle basic sets in terms of C_i. Both Theorems 1.2 and 1.3 easily follow from it. Set $I = \{0, 1, 2, \cdots, m\}$.

Proposition 3.1. *For any $i, j \in I$ with $i \neq j$, the following four conditions are equivalent to each other.*

(*a*) $\overline{C_i} \cap C_j \neq \emptyset$,
(*b*) $A(C_i) \cap W^s(\Lambda_j) \neq \emptyset$,
(*c*) $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) \neq \emptyset$,
(*d*) $W^s(\Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset$.

This proposition holds also for $i = j \geq 1$. It holds also for $i = j = 0$, if we set $\Lambda_0 = W^u(\Lambda_0) = \{[0 : 1 : 0]\}$.

proof. Since, for any $i \geq 1$, all the sets $\overline{C_i} \cap C_0, A(C_i) \cap W^s(\Lambda_0), \overline{W^s(\Lambda_i)} \cap W^s(\Lambda_0)$ and $W^s(\Lambda_i) \cap W^u(\Lambda_0)$ are empty, we may assume $j \geq 1$.

(*a*) ⇒ (*b*) Suppose $\overline{C_i} \cap C_j \neq \emptyset$. Take a sequence $x_n \in C_i$ tending to $x_0 \in C_j$. By Lemma 2.1, there exist $y \in \Lambda_j$ and k such that $f^k(x_0) \in W^s_{loc}(y)$. Hence, for any $n > 0$, there exists L_n such that $d(f^\ell(x_0), f^{\ell-k}(y)) < 1/n$ for $\ell \geq L_n$. For each fixed n, take k_n so that $d(f^{L_n}(x_{k_n}), f^{L_n}(x_0)) < 1/n$. Then it

follows $d(f^{L_n}(x_{k_n}), f^{L_n-k}(y)) < 2/n$. Since $f^{L_n-k}(y) \in \Lambda_j$, we conclude that $A(C_i) \cap \Lambda_j \neq \emptyset$.

$(b) \Rightarrow (c)$ It is evident since $A(C_i) \subset \overline{W^s(\Lambda_i)}$.

$(c) \Rightarrow (a)$ Suppose $W^s(\Lambda_i) \ni x_n \to x_0 \in W^s(\Lambda_j)$. Then there exist k and $q \in \Lambda_j$ such that $f^k(x_0) \in W^s_{loc}(q)$. Let U_q be the connected component of the vertical slice of $(J_p \times \mathbb{C}) \setminus J_2$ containing q. Then $U_q \supset W^s_{loc}(q)$ and, by Proposition 3.8 in [DH1], there exists $L > 0$ such that $f^L(U_q) = U_{f^L(q)}$ contains a critical point c, which is in C_j. Set $y = f^L(q)$. By Theorems 2.1 and 2.2, any compact subset in $U_y = f^{k+L}(U_{x_0})$ is approximated by compact subsets in $f^{k+L}(U_{x_n})$. Hence the branch of the critical locus through c must intersect $f^{k+L}(U_{x_n})$ for large n. Thus, for large n, there exist critical points $c_n \in f^{k+L}(U_{x_n})$ tending to c. Since $c_n \in C_i$, we conclude that $\overline{C_i} \cap C_j \neq \emptyset$. See Figure 1.

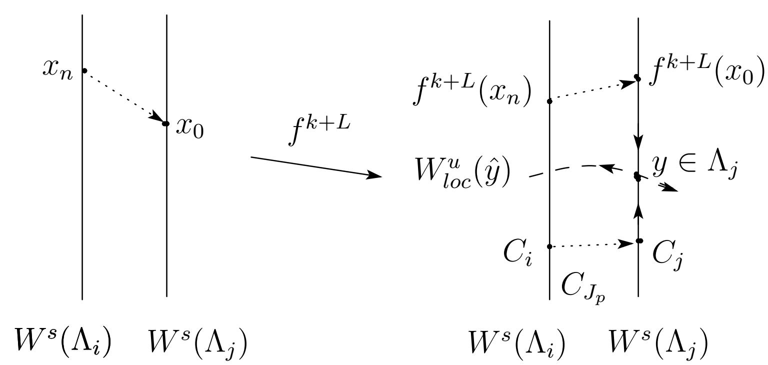

Figure 1: Local stable and unstable manifolds

$(c) \Rightarrow (d)$ Suppose $W^s(\Lambda_i) \ni x_n \to x_0 \in W^s(\Lambda_j)$. By the above argument, any compact subset in $U_y = U_{f^{k+L}(x_0)}$ is approximated by compact subsets in $U_{f^{k+L}(x_n)}$. Since, for any prehistory $\hat{y} \in \hat{\Lambda}_j$ of y, $W^u_{loc}(\hat{y})$ is transversal to the fiber, it follows that $U_{f^{k+L}(x_n)} \cap W^u_{loc}(\hat{y}) \neq \emptyset$ for large n. Thus we conclude $W^s(\Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset$.

$(d) \Rightarrow (c)$ We need a lemma.

Lemma 3.4. *Suppose $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) = \emptyset$. Then there exists $\delta > 0$ such that $W^s(\Lambda_i) \cap W^u_\delta(\hat{y}) = \emptyset$ for any $\hat{y} \in \hat{\Lambda}_j$.*

proof. Suppose, for any $n \geq 1$, there exist a prehistory $\hat{y}_n \in \hat{\Lambda}_j$ of $y_n \in \Lambda_j$ and $x_n \in W^s(\Lambda_i) \cap W^u_{1/n}(\hat{y}_n)$. Since the sequences $\{x_n\}$ and $\{y_n\}$ are bounded, there exist their convergent subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ respectively tending

to some points x_0 and y_0. Since $d(x_n, y_n) < 1/n$, we have $x_0 = y_0 \in \Lambda_j$. Thus we conclude $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) \neq \emptyset$. □

Now suppose $W^s(\Lambda_i) \cap W^u(\Lambda_j) \neq \emptyset$. If we take $p \in W^s(\Lambda_i) \cap W^u(\Lambda_j)$, there exists a prehistory $\hat{p} = (p_k)$ of p tending to Λ_j. Assume $\overline{W^s(\Lambda_i)} \cap W^s(\Lambda_j) = \emptyset$. Let $\delta > 0$ be the constant in Lemma 3.4. Then there exists $L > 0$ such that $d(p_k, \Lambda_j) < \delta$ for any $k \leq -L$, hence by Lemma 2.1, $p_{-L} \in W^u_\delta(\hat{y})$ for some $\hat{y} \in \hat{\Lambda}_j$. Since $p_{-L} \in W^s(\Lambda_i)$, we have $W^s(\Lambda_i) \cap W^u_\delta(\hat{y}) \neq \emptyset$, which contradicts Lemma 3.4. This completes the proof of Proposition 3.1. □

Set $I_i = \{j \in I; \overline{C_i} \cap C_j \neq \emptyset\}$.

Corollary 3.1. $A(C_i) \subset \cup_{j \in I_i} W^s(\Lambda_j) \quad (0 \leq i \leq m)$.

proof. By Theorem A, for $i \geq 1$ it follows $A(C_i) \subset W^u(\Lambda) \cap K_{J_p}$, which is, by Lemma E3.5 in [DH2], included in $W^s(\Lambda)$. In case $i = 0$, $A(C_0) = (A(C_0) \cap W^s(\Lambda_0)) \sqcup (A(C_0) \cap W^s(\Lambda))$ and $A(C_0) \cap W^s(\Lambda)$ is treated similarly. Now the assertion follows from Proposition 3.1. □

proof of Theorem 1.2. Note that C_i is closed if and only if $\overline{C_i} \cap C_j = \emptyset$ for any $j \neq i$. By Proposition 3.1, this is equivalent to $W^s(\Lambda_i) \cap W^u(\Lambda) = W^s(\Lambda_i) \cap W^u(\Lambda_i)$, which is, by Lemma 2.2, equal to Λ_i. By Theorem A and Proposition 3.1, it is also equivalent to $A(C_i) \subset W^s(\Lambda_i) \cap W^u(\Lambda) = \Lambda_i$. The inclusion $A(C_i) \supset \Lambda_i$ is trivial. This proves (3).

If C_i for some i is not closed, there exists $j \neq i$ such that $\overline{C_i} \cap C_j \neq \emptyset$. By Lemma 3.2, we have $A(C_i) \setminus \Lambda \neq \emptyset$, hence $A(C_{J_p}) \neq A_{pt}(C_{J_p})$. If C_i is closed for any i, then $A(C_{J_p}) = A_{pt}(C_{J_p})$ follows from (3). □

proof of Theorem 1.3. C_j is open in C_{J_p} if and only if $\overline{C_i} \cap C_j = \emptyset$ for any $i \neq j$. By Proposition 3.1 and Lemma 2.2, it is equivalent to $W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = W^u(\Lambda_j) \cap W^s(\Lambda_j) = \Lambda_j$. The equivalence of $W^u(\Lambda_j) \cap (J_p \times \mathbb{C}) = \Lambda_j$ and the continuity of the map $z \mapsto \Lambda_{j,z}$ in J_p can be proved by the same way as Theorem C. Continuity of the map $z \mapsto \Lambda_z$ is equivalent to that of the map $z \mapsto \Lambda_{j,z}$ for any $j \geq 1$. □

3.3 Continuity of the map $z \mapsto K_z$

By virtue of Proposition 3.1, we will characterize the properties in (5) and (6).

Theorem 3.1. *The following three properties are equivalent to each other.*
(a) C_0 *is closed,*
(b) the map $z \mapsto K_z$ *is continuous in* J_p*,*
(c) $A(C_{J_p}) = W^u(\Lambda) \cap W^s(\Lambda)$.

Recall that the continuity of a set-valued map decomposes into lower and upper semicontinuities. Theorem 3.1 reproves the equivalence (5) in Theorem C.

proof. Note that C_0 is closed if and only if $\overline{C_0} \cap (\cup_{j=1}^m C_j) = \emptyset$, which is, by Proposition 3.1, equivalent to $\overline{W^s(\Lambda_0)} \cap W^s(\Lambda) = \emptyset$ or to $W^s(\Lambda_0) \cap W^u(\Lambda) = \emptyset$.

$(a) \Leftrightarrow (b)$ Proposition 2.1 in [J3] says that $z \mapsto K_z$ is upper-semicontinuous and by Theorem 2.2, $z \mapsto J_z$ is continuous in J_p. Hence, $z \mapsto K_z$ is discontinuous at $z_0 \in J_p$ if and only if there exist a sequence z_n in J_p tending to z_0 and $w_0 \in int\, K_{z_0}$ such that $w_0 \notin K_{z_n}$ for any $n \geq 1$. That is, by Lemma 3.6 in [DH1], there exists a sequence (z_n, w_0) in $W^s(\Lambda_0)$ tending to a point $(z_0, w_0) \in W^s(\Lambda)$. This is equivalent to $\overline{W^s(\Lambda_0)} \cap W^s(\Lambda) \neq \emptyset$, that is, C_0 is not closed.

$(a) \Leftrightarrow (c)$ By Theorem A and Lemma E3.5 in [DH2], we have

$$A(C_{J_p}) = (W^u(\Lambda) \cap W^s(\Lambda)) \sqcup (W^u(\Lambda) \cap W^s(\Lambda_0)).$$

Thus C_0 is closed if and only if $A(C_{J_p}) = W^u(\Lambda) \cap W^s(\Lambda)$. $\Box$

The following is a direct consequence of Proposition 3.1 and (6).

Theorem 3.2. $W^u(\Lambda) \cap W^s(\Lambda) = \Lambda \Longleftrightarrow C_i$ *is closed for any* $i \geq 1$.

3.4 The sets $A(C_i)$ in general case

We have shown $A(C_i) = \Lambda_i$ if C_i is closed. In this subsection, we extend it and give a description of $A(C_i)$ in general case.

Theorem 3.3. *Recall that* $I_i = \{j \in I; \overline{C_i} \cap C_j \neq \emptyset\}$. *Then we have*

$$A(C_i) \subset (\cup_{j \in I_i} W^u(\Lambda_j)) \cap (\cup_{j \in I_i} W^s(\Lambda_j)).$$

If C_i is closed, $I_i = \{i\}$. Then Theorem 3.3 says $A(C_i) \subset W^u(\Lambda_i) \cap W^s(\Lambda_i) = \Lambda_i$. Thus Theorem 3.3 generalizes Theorem 1.2.

proof. By virtue of Corollary 3.1, we have only to show $A(C_i) \subset \cup_{j \in I_i} (W^u(\Lambda_j) \cap (J_p \times \mathbb{C}))$. We give a proof mainly for the case $i \geq 1$. The case $i = 0$ can be done with a minor change and will be given at the end. Below, we repeatedly use the argument in the proof of Lemmas 3.2 and 3.3.

Suppose $p \in A(C_i)$. Then there exists $x_n \in C_i$ and $m_n \nearrow \infty$ such that $f^{m_n}(x_n) \to p$. If $p \in \Lambda$, by Corollary 3.1, we have $j \in I_i$ and $p \in \cup_{j \in I_i} \Lambda_j$. This holds also for $i = 0$.

In the sequel, we assume $p \notin \Lambda$. We may assume $p \notin \overline{\cup_{j=1}^m U_j}$, where U_j is a small open neighborhood of Λ_j. We may also assume $x_n \to x_0$. If $x_0 \in C_i$, by Lemma 3.3, we have $p \in \Lambda_i$, which contradicts $p \notin \Lambda$. Hence $x_0 \in C_{i_1}$ for some $i_1 \neq i$ and by Proposition 3.1, $i_1 \in I_i$. As in the proof of Lemma 3.2, we take K_1 so that $f^k(x_0) \in U_{i_1}$ for $k \geq K_1$ and set $k_n^{(1)} = min\{k \geq K_1; f^k(x_n) \notin U_{i_1}\}$. If $m_n < k_n^{(1)}$ for infinitely many n, then $p = \lim f^{m_n}(x_n) \in \overline{U_{i_1}}$, a contradiction. Thus $m_n \geq k_n^{(1)}$ for large n. We may assume $f^{k_n^{(1)}}(x_n)$ tends to some $y^{(1)} \in W^u(\Lambda_{i_1}) \setminus \Lambda$. Suppose $y^{(1)} \in W^s(\Lambda_{i_2})$. Since $y^{(1)} \in A(C_i)$, we have $i_2 \in I_i$.

Now take K_2 so that $f^k(y^{(1)}) \in U_{i_2}$ for $k \geq K_2$ and $k_n^{(2)} = min\{k \geq k_n^{(1)} + K_2; f^k(x_n) \notin U_{i_2}\}$. If $k_n^{(1)} \leq m_n \leq k_n^{(1)} + K_2$ for infinitely many n, there exists $j \leq K_2$ so that $m_n = k_n^{(1)} + j$ for infinitely many n. Then we have

$$p = \lim f^{m_n}(x_n) = \lim f^{k_n^{(1)}+j}(x_n) = f^j(y^{(1)}) \in W^u(\Lambda_{i_1}).$$

Otherwise, we have $m_n \geq k_n^{(1)} + K_2$ for large n. We may assume $f^{k_n^{(2)}}(x_n) \to y^{(2)} \in W^u(\Lambda_{i_2}) \setminus \Lambda$. If $m_n < k_n^{(2)}$ for infinitely many n, then $p = \lim f^{m_n}(x_n) \in \overline{U_{i_2}}$, a contradiction. Thus $m_n \geq k_n^{(2)}$ for large n.

Repeating this argument, we eventually meet Λ_i. That is, there exist ℓ and $i_j \in I_i, 1 \leq j \leq \ell$ such that

$$\Lambda_{i_1} \succ \Lambda_{i_2} \succ \cdots \succ \Lambda_{i_\ell} = \Lambda_i.$$

Suppose $k_n^{(\ell)} < \infty$ for infinitely many n. Then, further repeating this argument, we must meet Λ_i again. That is, there exists a sequence :

$$\Lambda_{i_1} \succ \cdots \succ \Lambda_{i_\ell} = \Lambda_i \succ \cdots \succ \Lambda_i.$$

This contradicts Theorem 2.3. Thus we conclude that, for large n, $k_n^{(\ell)} = \infty$ and $f^k(x_n) \in U_{i_\ell}$ for $k \geq k_n^{(\ell-1)} + K_\ell$. Since $p \notin \overline{U_{i_\ell}}$, we may conclude $k_n^{(\ell-1)} \leq m_n \leq k_n^{(\ell-1)} + K_\ell$ for large n. Then, there exists $j \leq K_\ell$ such that $m_n = k_n^{(\ell-1)} + j$ for infinitely many n and

$$p = \lim f^{m_n}(x_n) = \lim f^{k_n^{(\ell-1)}+j}(x_n) = f^j(y^{(\ell-1)}) \in W^u(\Lambda_{i_{\ell-1}}).$$

Thus we conclude $A(C_i) \subset \cup_{j \in I_i} (W^u(\Lambda_j) \cap (J_p \times \mathbb{C}))$.

Let us consider the case $i = 0$. Let $p \in A(C_0)$. The argument as above works as long as $y^{(\ell)} \in W^s(\Lambda)$. Suppose $y^{(\ell)} \notin W^s(\Lambda)$. Then it belongs to $W^s(\Lambda_0)$. Since $W^s(\Lambda_0)$ is open, for large n, $f^{k_n^{(\ell)}}(x_n)$ is contained in a neighborhood of $y^{(\ell)}$ in $W^s(\Lambda_0)$.

If $m_n \le k_n^{(\ell)}$ for infinitely many n, then $p \in \overline{U_{i_\ell}}$, a contradiction. Thus $m_n \ge k_n^{(\ell)}$ for large n. Now suppose the sequence $\{m_n - k_n^{(\ell)}\}$ is unbounded. By taking a subsequence tending to ∞, the sequence $f^{m_n}(x_n) = f^{m_n - k_n^{(\ell)}} \circ f^{k_n^{(\ell)}}(x_n)$ tends to ∞, which is a contradiction. Hence $0 < m_n - k_n^{(\ell)} < K_{\ell+1}$ for some $K_{\ell+1} > 1$. Then there exists $j \ge 1$ such that $m_n = k_n^{(\ell)} + j$ for infinitely many n and $p = \lim f^{m_n}(x_n) = f^j(y^{(\ell)}) \in W^u(\Lambda_{i_\ell})$. This completes the proof. □

As a corollary, we extend the equivalence (3) in Theorem 1.2 to a union of saddle basic sets. Let I' be a subset of I and set $\Lambda' = \cup_{j\in I'}\Lambda_j$, $C' = \cup_{j\in I'}C_j$.

Proposition 3.2. *$A(C') \subset W^u(\Lambda') \cap W^s(\Lambda') \iff C'$ is closed.*

proof. ($\Leftarrow$) Suppose C' is closed. Then, for any $k \in I'$, it follows $I_k \subset I'$ since $\overline{C_k} \subset \overline{C'} = C'$. By Theorem 3.3, for any $k \in I'$, we have

$$A(C_k) \subset (\cup_{j\in I_k} W^u(\Lambda_j)) \cap (\cup_{j\in I_k} W^s(\Lambda_j)) \subset W^u(\Lambda') \cap W^s(\Lambda').$$

Hence we have $A(C') \subset W^u(\Lambda') \cap W^s(\Lambda')$.

($\Rightarrow$) If C' is not closed, there exists $k \in I'$ and $i \notin I'$ such that $\overline{C_k} \cap C_i \neq \emptyset$. Then, by Proposition 3.1, it follows $A(C_k) \cap W^s(\Lambda_i) \neq \emptyset$. Thus we conclude that $A(C') \not\subset W^u(\Lambda') \cap W^s(\Lambda')$. This completes the proof. □

We do not know if the equality holds in Proposition 3.2. After Proposition 3.2, a partition $I = \sqcup_i \tilde{I}_i$ of I such that $\cup_{j\in\tilde{I}_i} C_j$ is closed for any i will give equality. Define the equivalence relation among I_i generated by $I_i \cap I_j \neq \emptyset$. That is, $I_i \sim I_j$ if there exist $i = i_1, i_2, \cdots, i_k = j$ in I such that $I_{i_\ell} \cap I_{i_{\ell+1}} \neq \emptyset$ for $1 \le \ell \le k-1$. Let $I = \sqcup_i \tilde{I}_i$ be the partition of I such that each $\tilde{I}_i$ is the union of I_j in the same equivalence class. Then $I_j \subset \tilde{I}_i$ if $j \in \tilde{I}_i$ and it follows that the set $\cup_{j\in\tilde{I}_i} C_j$ is closed for each i. By Proposition 3.2, for each i, we have

$$\cup_{j\in\tilde{I}_i} A(C_j) \subset (\cup_{j\in\tilde{I}_i} W^u(\Lambda_j)) \cap (\cup_{j\in\tilde{I}_i} W^s(\Lambda_j)). \tag{7}$$

Since $A(C_{J_p}) = \cup_i \cup_{j\in\tilde{I}_i} A(C_j)$, the equality holds in (7). Thus we have

Proposition 3.3. $\cup_{j\in\tilde{I}_i} A(C_j) = (\cup_{j\in\tilde{I}_i} W^u(\Lambda_j)) \cap (\cup_{j\in\tilde{I}_i} W^s(\Lambda_j))$.

3.5 Proof of Theorem 1.4

We use the stability of hyperbolic sets under perturbation established in [J2, DH1, DH2]. Consider a holomorphic family $\{f_a; a \in \mathbb{D}\}$ of Axiom A polynomial skew products containing $f = f_0$. The holomorphic motion $\Psi(a, z, w) = (\varphi_a(z), \psi_a(z, w))$ of $J_2 = J_2(f)$ gives homeomorphisms

$$\varphi_a : J_p \to J_{p_a}, \quad \psi_a(z, \cdot) : J_z(f) \to J_{\varphi_a(z)}(f_a).$$

We have a holomorphic motion $\hat{h}_a : \hat{\Lambda} := \hat{\Lambda}(f) \to \hat{\Lambda}_a := \hat{\Lambda}(f_a)$ of $\hat{\Lambda}(f)$. As for $\Lambda := \Lambda(f)$, we only have a surjective continuous map $h_a : \hat{\Lambda} \to \Lambda_a := \Lambda(f_a)$, which is induced from $\hat{h}_a$ and depends holomorphically on a. Hence, for any $z \in J_p$, the map $a \mapsto \Lambda_{a,\varphi_a(z)} := \Lambda_a \cap (\{\varphi_a(z)\} \times \mathbb{C})$ is continuous. Note that the same holds also for each saddle basic set Λ_i of f and that the map $a \mapsto J_{\varphi_a(z)}(f_a)$ is continuous.

Suppose a critical point $x = (z, w) \in C_{J_p}$ of f lies in C_i. Then there exist $y = (u, v) \in \Lambda_i$ and $n \geq 0$ such that $x_n = f^n(x) \in W^s_{loc}(y)$. Then U_{x_n}, the connected component of the vertical slice of $(J_p \times \mathbb{C}) \setminus J_2$ containing x_n, contains y because $W^s_{loc}(y)$ lies in a vertical fiber. Set $z_a = \varphi_a(z)$ and let $x_a = (z_a, w_a)$ be a nearby critical point of f_a. If a is small, $x_{a,n} = f_a^n(x_a)$ is close to $f^n(x)$. By the continuity of the maps $a \mapsto J_{z_a}(f_a)$ and $a \mapsto \Lambda_{a,z_a}$, the point $y_a = h_a(\hat{y})$ for any prehistory $\hat{y}$ of y lies in $U_{x_{a,n}} \cap \Lambda_{a,i}$. Hence it follows $x_a \in W^s(\Lambda_{a,i})$ for small a. Theorem 2.1 says that the postcritical set is disjoint from the second Julia set. By the above continuity, we conclude that $x_a \in W^s(\Lambda_{a,i})$ holds as long as the holomorphic motion exists.

Now suppose $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ for $f = f_0$. For any $C_a \in \mathcal{C}(C_{J_{p_a}})$, there exists a connected component J_a of J_{p_a} such that C_a is a connected component of C_{J_a}. Then $J_a = \varphi_a(J)$ for some connected component J of J_p and there exists $C \in \mathcal{C}(C_{J_p})$ close to C_a, which is a connected component of C_J. By Theorem 1.1, we have $C \subset C_i$ for some i. There exists n such that, for any $x \in C$, $f^n(x) \in W^s_{loc}(y)$ for some $y \in \Lambda_i$. Thus we can apply the above argument simultaneously to any $x \in C$ and we conclude that $C_a \subset C_{a,i} := C_{J_{p_a}} \cap W^s(\Lambda_{a,i})$, hence $A_{cc}(C_{J_p}) = A_{pt}(C_{J_p})$ holds for $f = f_a$ as long as the holomorphic motion exists.

Next suppose $A(C_{J_p}) = A_{pt}(C_{J_p})$ for $f = f_0$. By Theorem 1.2, $A(C_i) = \Lambda_i$ holds for any i. There exists n such that, for any $x \in C_i$, $f^n(x) \in W^s_{loc}(y)$ for some $y \in \Lambda_i$. Then nearby critical point of f_a for small a belongs to $W^s(\Lambda_{a,i})$. By Theorem 1.2, all C_i are closed, hence mutually disjoint compact sets. Thus so are $C_{a,i}$ and $A(C_{J_p}) = A_{pt}(C_{J_p})$ holds also for $f = f_a$ for small a. By Theorem 2.1, this holds as long as the holomorphic motion exists.

Since any maps in the same hyperbolic component are connected by a chain of disks where the holomorphic motion exists, we get the conclusion. □

3.6 Proof of Theorem 1.5

First we estimate the component-wise accumulation set $A_{cc}(C_{J_p})$. The key tool is a result due to Qiu & Yin [QY] that any non-point component of a disconnected polynomial Julia set is preperiodic. Although a complete char-

acterization of the set $A_{cc}(C_{J_p})$ is not known, our estimate is enough to prove Theorem 1.5. Let J_{per} be the union of the periodic non-point components of J_p. Then $J_{per} = J_p$ if J_p is connected and $J_{per} = \emptyset$ if J_p is a Cantor set.

Proposition 3.4. $A_{cc}(C_{J_p}) \subset \Lambda \cup (W^u(\Lambda) \cap (J_{per} \times \mathbb{C}))$.

proof. For any $C \in \mathcal{C}(C_{J_p})$, there exists a connected component J_0 of J_p such that C is a connected component of C_{J_0}. If J_0 is a point component, then C_{J_0} consists of finitely many points, hence $A(C_{J_0}) \subset \Lambda$. Otherwise, it is a preimage of a periodic component J_1 of J_p, i.e. $p^k(J_0) = J_1 \subset J_{per}$ for some k. See [QY], Theorem in Section 5. Let J_{-k} be the (finite) union of the connected components of $p^{-k}(J_{per})$. We have only to show that $\overline{\cup_{k \geq 0} A(C_{J_{-k}})} \subset W^u(\Lambda) \cap (J_{per} \times \mathbb{C})$. By Theorem 2.1, the set $X = \overline{\cup_{k \geq 0} f^k(C_{J_{-k}})}$ is disjoint from J_2. Then Proposition 3.3 in [DH1] says $A(X) \subset W^u(\Lambda) \cap (J_p \times \mathbb{C})$. Evidently we have $A(X) \subset J_{per} \times \mathbb{C}$. Thus it follows

$$\overline{\cup_{k \geq 0} A(C_{J_{-k}})} = \overline{\cup_{k \geq 0} A(f^k(C_{J_{-k}}))} \subset A(X) \subset W^u(\Lambda) \cap (J_{per} \times \mathbb{C}).$$

This completes the proof. $\square$

We investigate the slice $W^u(\Lambda)_z$ of $W^u(\Lambda)$ at $z \in J_p$.

Lemma 3.5. *Suppose* $\Lambda \neq \emptyset$. *Then* $W^u(\Lambda)_z \neq \emptyset$ *for any* $z \in J_p$.

proof. If $\Lambda \neq \emptyset$, there exists a saddle periodic point $x = (z_0, w_0) \in \Lambda$. Let $\hat{x} \in \hat{\Lambda}$ be any one of the prehistories of x. Recall that the set $\{p^{-k}(z); k \geq 0\}$ is dense in J_p for any $z \in J_p$. Then, for any $z \in J_p$, there exist sequences $n_k \nearrow \infty$ and $z_{-k} \in p^{-n_k}(z)$ tending to z_0. Since $W^u_{loc}(\hat{x})$ is transversal to the vertical fiber, for large k, there exists w_{-k} such that $x_{-k} := (z_{-k}, w_{-k}) \in W^u_{loc}(\hat{x}) \cap (J_p \times \mathbb{C})$. Then $(z, w) := f^{n_k}(x_{-k}) \in W^u(\Lambda) \cap (J_p \times \mathbb{C})$. This completes the proof. $\square$

By virtue of Proposition 3.4, we get the following.

Proposition 3.5. *Suppose* J_p *is disconnected and* $\Lambda \neq \emptyset$. *If* $A_{cc}(C_{J_p}) = A(C_{J_p})$, *then* $W^u(\Lambda)_z = \Lambda_z$ *for* $z \in J_p \setminus J_{per}$ *and* $\Lambda_z \neq \emptyset$ *for any* $z \in J_p$.

proof. Suppose $A_{cc}(C_{J_p}) = A(C_{J_p})$. Then, from Proposition 3.4, we have $W^u(\Lambda) \cap (J_p \times \mathbb{C}) \subset \Lambda \cup (W^u(\Lambda) \cap (J_{per} \times \mathbb{C}))$. Hence it follows $W^u(\Lambda)_z = \Lambda_z$ for $z \in J_p \setminus J_{per}$, which is not empty by Lemma 3.5.

Note that $J_p \setminus J_{per}$ is dense in J_{per}. In fact, otherwise, a point in J_{per} has a neighborhood U disjoint from $J_p \setminus J_{per}$. Then $\cup_{n \geq 0} p^n(U)$ never intersects $J_p \setminus J_{per}$, a contradiction. Thus, for any $z \in J_{per}$, there exists $z_n \in J_p \setminus J_{per}$

tending to z. Take $(z_n, w_n) \in \Lambda$. Since $\{w_n\}$ is a bounded set, there is a subsequence $\{w_{n_k}\}$ converging to a point w. Then $(z, w) = \lim_{k\to\infty}(z_{n_k}, w_{n_k}) \in \Lambda$. This completes the proof. $\square$

Let $d_H(\cdot, \cdot)$ be the Hausdorff distance of compact sets in $\mathbb{C}$. Set $\mathbb{D}(z_0, r) = \{z \in \mathbb{C}; |z - z_0| < r\}$.

Proposition 3.6. *Suppose J_p is disconnected, $\Lambda \neq \emptyset$ and $W^u(\Lambda)_z = \Lambda_z$ for any $z \in J_p \setminus J_{per}$. Let z_0 be any point in J_p. Then, for any $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_H(\Lambda_z, \Lambda_{z_0}) < \epsilon \quad \text{for any} \quad z \in \mathbb{D}(z_0, \delta) \cap (J_p \setminus J_{per}). \tag{8}$$

In particular, the map $z \mapsto \Lambda_z$, restricted to $J_p \setminus J_{per}$, is continuous.

proof. We prove the proposition by contradiction. Suppose there exist an $\epsilon > 0$ and a sequence $z_n \in J_p \setminus J_{per}$ tending to $z_0 \in J_p$ with $d_H(\Lambda_{z_n}, \Lambda_{z_0}) \geq \epsilon$. Since Λ is closed, the map $z \mapsto \Lambda_z$ is upper semicontinuous on J_p. Thus it follows that Λ_{z_0} does not lie in the ϵ-neighborhood of Λ_{z_n}, that is, there exists $w_n \in \Lambda_{z_0}$ such that $\Lambda_{z_n} \cap \mathbb{D}(w_n, \epsilon) = \emptyset$. We may assume $w_n \to w_0 \in \Lambda_{z_0}$. Then $\Lambda_{z_n} \cap \mathbb{D}(w_0, \epsilon/2) = \emptyset$ for large n. Since $W^u_{loc}(\hat{x})$ is transversal to the fiber for any prehistory $\hat{x}$ of $x = (z_0, w_0) \in \Lambda$, we have $W^u(\Lambda)_{z_n} \cap \mathbb{D}(w_0, \epsilon/2) \neq \emptyset$ for large n. This is a contradiction since $W^u(\Lambda)_{z_n} = \Lambda_{z_n}$. $\square$

Now Theorem 1.5 follows from the next theorem.

Theorem 3.4. *Suppose J_p is disconnected. If $W^u(\Lambda)_z = \Lambda_z$ for any $z \in J_p \setminus J_{per}$, then the map $z \mapsto \Lambda_z$ is continuous in J_p. In particular, if $A_{cc}(C_{J_p}) = A(C_{J_p})$, then $A_{pt}(C_{J_p}) = A_{cc}(C_{J_p}) = A(C_{J_p})$.*

proof. Theorem is true if $\Lambda = \emptyset$. Hence we may assume $\Lambda \neq \emptyset$. Take a point z_0 in J_p. We have only to show (8) also for $z \in \mathbb{D}(z_0, \delta) \cap J_{per}$. For any $\epsilon > 0$, take $\delta > 0$ as in Proposition 3.6. Now, for any $z \in \mathbb{D}(z_0, \delta) \cap J_{per}$, there exists a sequence $\{z_n\}$ in $J_p \setminus J_{per}$ tending to z. Applying Proposition 3.6 to $z_0 = z$, it follows $\Lambda_{z_n} \to \Lambda_z$. Note that $z_n \in \mathbb{D}(z_0, \delta)$ for large n. Again by Proposition 3.6, we have $d_H(\Lambda_{z_n}, \Lambda_{z_0}) < \epsilon$ for such n, hence $d_H(\Lambda_z, \Lambda_{z_0}) \leq \epsilon$. This implies the desired continuity on J_{per}. The last statement follows from Theorem C. This completes the proof. $\square$

3.7 Proof of Theorem 1.6

The proof of Theorem 1.6 is a higher degree analogue of that of Theorem 6.1 in [DH1]. Consider the degree four polynomial skew product $f(z, w) =$

$(p(z), q(z,w))$, where $q(z,w) = w^4 + 4(2-z)$ and p is a *real biquadratic polynomial* of the form :

$$p(z) = p_{a,b}(z) = (z^2+a)^2 + b, \quad (a,b) \in \mathbb{R}^2.$$

First we investigate the dynamics of p so that we find maps with one critical point escaping while the others tending to an attracting cycle. Next we show that the map f has the desired property. For our maps, the saddle set Λ consists of a single saddle fixed point. Then exactly one critical point tends to this point while all the others escape to infinity. The hard part is to show that $D_{J_p} \cap J_2 = \emptyset$. Most of this subsection, Lemmas 3.11 - 3.15, is devoted to control the behavior of the critical orbits.

Here we give a brief summary on the external rays of a monic polynomial p of degree d. See Milnor [Mil] for the details. Let φ_p be the *Böttcher coordinate* of p. It is a conformal map defined in a neighborhood of ∞ conjugating p to the map $z \mapsto z^d$. Then the *external ray* $R_p(\theta)$ of p with angle θ is defined by $R_p(\theta) = \varphi_p^{-1}(\{re^{2\pi i\theta}; r > r_p\})$ for some $r_p \geq 1$ depending on p. It maps $R_p(\theta)$ to $R_p(d\theta)$. Hence, it is continued until it meets a critical point. If it is continued to $r > 1$ and the limit $z = \lim_{r\to 1} \varphi_p^{-1}(re^{2\pi i\theta})$ exists, it is said to land at z. A fundamental fact is as follows : for $\theta \in \mathbb{Q}$, the θ-ray $R_p(\theta)$ lands at some point $z \in J_p$ unless it meets a point in the backward orbit of a critical point. If the 0-ray lands, its landing point is a fixed point, which we call the β-fixed point of p and denote by β_p.

The map $p = p_{a,b}$ has three critical points : $z = 0, \pm\sqrt{-a}$ and two critical values : $p(0) = a^2 + b$, $p(\pm\sqrt{-a}) = b$. Note that critical values are always real. The *connectedness locus* $\mathcal{C}$ of this family is the set of parameters (a,b) in $\mathbb{R}^2$ so that the Julia set $J(p_{a,b})$ is connected, or equivalently, all the critical points of $p_{a,b}$ have bounded orbits. It is described as follows. See Figure 2. The dark region indicates $\mathcal{C}$.

Lemma 3.6. *The boundary of $\mathcal{C}$ consists of the following three curves :*

$Per_1^+(1) : p'(\beta_p) = 1 : (a,b) = (\frac{1}{2t} - \frac{t^2}{4}, \frac{t}{2} - \frac{1}{4t^2}),\ \frac{1}{\sqrt[3]{4}} \leq t \leq \sqrt[3]{4},$

$Preper_{(1)1} : p(0) = \beta_p : b = -a^2 + \sqrt{-2a},\ -2 \leq a \leq -\frac{\sqrt[3]{2}}{4},$

$Preper_{(2)1} : p(b) = \beta_p = -b : a = -b^2 + \sqrt{-2b},\ -2 \leq b \leq -\frac{\sqrt[3]{2}}{4}.$

proof. Since the critical values are real, we consider the dynamics of p only on the real axis. For large b, the graph of $y = p(x)$ sits above the diagonal line $y = x$, hence the orbits of all real points tend to $+\infty$. Decreasing b, we meet

a parameter at which the graph of $y = p(x)$ is tangent to the line $y = x$ at β_p. This parameter lies on $Per_1^+(1)$, where β_p is a parabolic fixed point with multiplier one. Below this locus, the map p has at least two real fixed points. The point β_p is the largest one. Then $p \in \mathcal{C}$ if and only if $K_p \cap \mathbb{R} = [-\beta_p, \beta_p]$. If $a < 0$, then p has the local maximum $p(0)$ and the local minimum b. Hence $p \in \mathcal{C}$ if and only if $-\beta_p \leq b$ and $p(0) \leq \beta_p$. In case $a \geq 0$, since p has a unique local minimum $p(0) = a^2 + b \geq b$, $p \in \mathcal{C}$ if and only if $b \geq -\beta_P$.

The locus $Preper_{(1)1} : p(0) = \beta_p$ is included in the set : $\{(a^2+b)^2+a\}^2+b = a^2+b$, that is, $(a^2+b)^2+a = \pm a$. Since $(a^2+b)^2+a = a$ implies $p(0) = 0$, $Preper_{(1)1}$ is included in $(a^2+b)^2 = -2a$ i.e. $b = -a^2 \pm \sqrt{-2a}$. Since $p(0) = a^2+b = \beta_p > 0$, we conclude that $Preper_{(1)1}$ is written as $b = -a^2 + \sqrt{-2a}$.

The locus $Preper_{(2)1} : b = -\beta_p$ or $p(b) = -b$ is included in $(b^2+a)^2+2b = 0$, that is, $b \leq 0$ and $a = -b^2 \pm \sqrt{-2b}$. On the curve $a = -b^2 - \sqrt{-2b}$, we have

$$\begin{aligned} p(x) - x &= x^4 - 2(b^2 + \sqrt{-2b})x^2 - x + b^4 + 2b^2\sqrt{-2b} - b \\ &= (x+b)(x^3 - bx^2 - (b^2 + 2\sqrt{-2b})x + b^3 + 2b\sqrt{-2b} - 1) \\ &=: (x+b)g(x), \end{aligned}$$

and $g(-b) = 4b\sqrt{-2b} - 1 < 0$. Thus p has a fixed point larger than $-b$. That is, $\beta_p > -b$, hence $Preper_{(2)1}$ is written by $a = -b^2 + \sqrt{-2b}$. This completes the proof. □

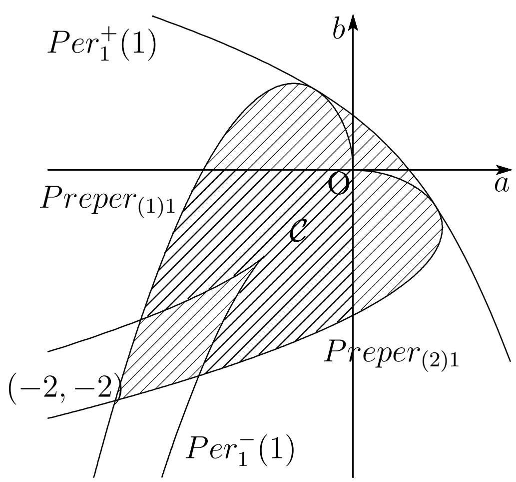

Figure 2: Parameter space

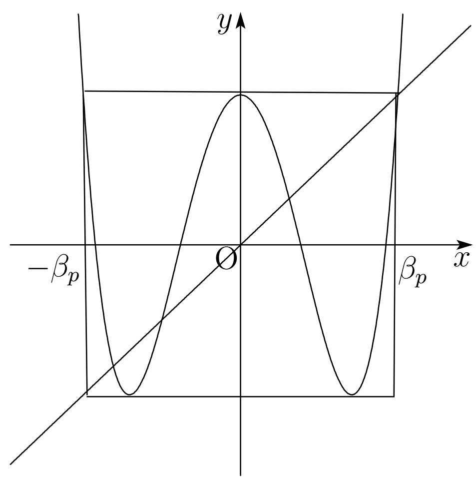

Figure 3: Graph of $p_{-2,-2}$

We consider the subfamily $Preper_{(1)1}$. Note that $p_{-2,-2}$ is the second iterate of the Chebyshev polynomial $z^2 - 2$ of degree two.

Lemma 3.7. *There exists a sequence* $(a_n, b_n) \in Preper_{(1)1}$ *tending to* $(-2, -2)$ *such that* $\sqrt{-a_n}$ *is a superattracting periodic point of* $\tilde{p}_n := p_{a_n, b_n}$ *of period* n.

proof. For $(a, b) \in Preper_{(1)1}$, take an increasing sequence $x_n \in p^{-n}(\sqrt{-a})$ in $\mathbb{R}$ tending to β_p. Then there exists a unique n such that $x_{n-1} \leq p^2(\sqrt{-a}) < x_n$. Moving along $Preper_{(1)1}$, we get a parameter (a_n, b_n) satisfying $p^2(\sqrt{-a}) = x_{n-1}$, that is $p^{n+1}(\sqrt{-a}) = \sqrt{-a}$. If the parameter approaches $(-2, -2)$, $p^2(\sqrt{-a})$ tends to β_p, hence it becomes larger than x_n for arbitrarily large n. This completes the proof. $\square$

By Lemma 3.6, a small perturbation from $Preper_{(1)1}$ leaves the connectedness locus. Thus, we get a small perturbation p_n of $\tilde{p}_n$ outside the connectedness locus. For p_n, 0 is escaping and the other critical points $\pm\sqrt{-a}$ belong to the basin of a superattracting n-cycle of p_n. Thus $p = p_n$ satisfies the property (a) in Theorem 1.6. See Figures 3 - 5.

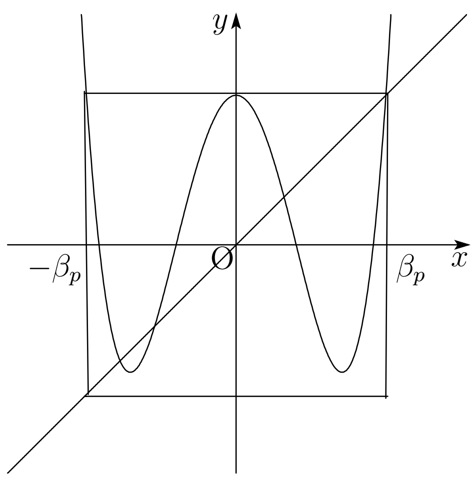

Figure 4: Graph of $\tilde{p}_n$

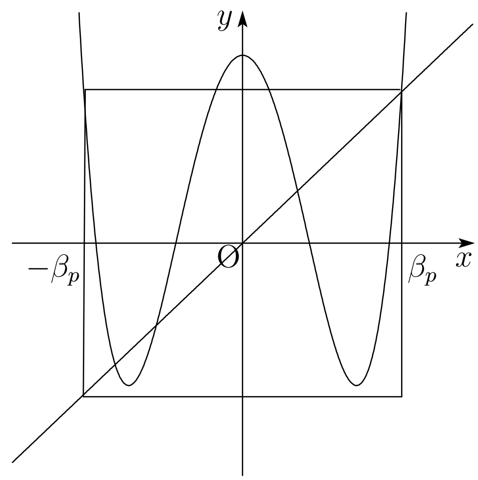

Figure 5: Graph of p_n

By upper-semicontinuity of the filled-in Julia set, we have the following.

Lemma 3.8. *There exists a sequence* ϵ_n *tending to* 0 *such that*
$K_{p_n} \subset [-5/2, 5/2] \times [-\epsilon_n, \epsilon_n]$.

Now we will control the behaviour of the critical orbits.

Lemma 3.9. *There exists* n_0 *such that for any* $n \geq n_0$ *and for any* $z \in K_{p_n}$, *we have*
(a) $q_z(\mathbb{C} \setminus \mathbb{D}(0, 5/2)) \subset \mathbb{C} \setminus \mathbb{D}(0, 35/2)$,
(b) $K_z \subset \mathbb{D}(0, 5/2)$.

proof. By Lemma 3.8, there exists n_0 such that $\epsilon_n \leq 1/4$ for $n \geq n_0$. Then $|2-z| \leq 5$ and, if $|w| \geq 5/2$, we have

$$|q_z(w)| \geq |w|^4 - 4|2-z| \geq |w|(|w|^3 - \frac{20}{|w|}) \geq (\frac{125}{8} - 8)|w| \geq 7|w| \geq \frac{35}{2}.$$

This completes the proof. □

Set $f_n(z,w) = (p_n(z), q(z,w))$ and $f_n^k(z,w) = (p_n^k(z), Q_{n,z}^k(w))$.

Lemma 3.10. *For $n \geq n_0$, $D_{A_{p_n}} \cap J_{A_{p_n}} = \emptyset$.*

proof. Note that $A_{p_n} \subset \mathbb{R}$ and $C_{A_{p_n}} = A_{p_n} \times \{0\}$. Take $x \in A_{p_n}$ and set $(x_k, y_k) = f_n^k(x, 0)$. Then $x_k = p_n^k(x)$. Recall that an attracting cycle contains the critical value $b = p_n(\pm\sqrt{-a})$ in the cycle of its immediate basins. Since $b = b_n < 0$, the basin U containing b satisfies $U \cap \mathbb{R} \subset \{\operatorname{Re} z < 0\}$. Hence $x_j < 0$ for some $0 \leq j \leq n-1$. Then $y_{j+1} = y_j^4 + 4(2-x_j) \geq 8$. By Lemma 3.9, $y_{j+1} \notin K_{x_{j+1}}$, hence $y_k \notin K_{x_k}$ for any k. Thus we conclude $D_{A_{p_n}} \cap J_{A_{p_n}} = \emptyset$. This completes the proof. □

In the following lemma, the constant r is fixed. Later we assume $r < 7/128$. Set $B(z_0, r) = \{z \in \mathbb{C}; |z - z_0| \leq r\}$.

Lemma 3.11. *For any fixed $r > 0$, there exists $N > 0$ and $n_1 \geq n_0$ such that for any $n \geq n_1$ and for any $z \in J_{p_n} \setminus B(2, r)$, there exists $0 \leq j < N$ with $\operatorname{Re} z_j \leq 1$.*

proof. For $p_\infty := p_{-2,-2}$, the $\pm 1/4$-rays land at 0 and the $\pm 3/16$-rays land at $x_\infty = \sqrt{2-\sqrt{2}} = 0.7655\ldots$, a preimage of 0. Thus the set of landing points $z \in J_{p_\infty}$ of external rays with angles in $[3/16, 13/16]$ sits in $\operatorname{Re} z \leq 1$. We will show the same property also for p_n for large n.

First we consider maps $\tilde{p}$ on $Preper_{(1)1}$. The 0-ray lands at $\beta_{\tilde{p}}$. By the symmetry of the filled-in Julia set, the 1/4-ray and the $-1/4$-ray are respectively the positive and negative imaginary axes. As their preimages, the $\pm 3/16$-rays and their landing points depend continuously on $\tilde{p} \in Preper_{(1)1}$. By the upper-semicontinuity of the filled-in Julia sets, we conclude that the same property holds for $\tilde{p}_n$ for large n.

Recall that p_n is obtained by a small perturbation of $\tilde{p}_n$ outside the connectedness locus. Let p be such a perturbation of $\tilde{p} \in Preper_{(1)1}$. It is easy to see that, for p, the 0-ray also lands at β_p. Again by the symmetry, the positive and negative imaginary axes respectively form the 1/4- and $-1/4$-rays for p and meet at the escaping critical point 0. As their preimages, the 3/16- and

$-3/16$-rays in the upper and lower half plane respectively meet at a preimage of 0 and depend continuously on p. By the same way as above, we conclude that the set of points $z \in J_{p_n}$ with external angles in $[3/16, 13/16]$ sits in $\operatorname{Re} z \leq 1$ for large n.

Note that the part of K_{p_∞} on the right side of the $\pm 3/4^j$-rays sits in $B(2, r)$ for large j. The same holds for p sufficiently close to p_∞, since the $\pm 3/4^j$-rays depend continuously on the parameter.

Now define a map $m : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ by $m(t) = 4t$. Then $m^{-1}((3/16, 13/16))$ contains $(3/64, 13/64) \cup (51/64, 61/64)$ and the sum of the preimages $\cup_{j\geq 2}\{(3/4^j, 13/4^j) \cup (-13/4^j, -3/4^j)\}$ covers $\mathbb{R}/\mathbb{Z}$ except the angle $\{0\}$. The part of K_{p_n} on the right side of the $\pm 3/4^j$-rays is included in $B(2, r)$ for large j. This completes the proof. $\square$

Note that, the β-fixed point $\beta_n := \beta_{p_n}$ forms a single point component of J_{p_n} because the $\pm 3/4^j$-rays for $j \geq 2$ separate β_n from any other points in J_p.

Lemma 3.12. *Let N be given in Lemma 3.11. There exists $n_2 \geq n_1$ and $\delta > 0$ such that*

$$Q^N_{n,z}(\{|\operatorname{Im} w| < \delta\}) \cap \mathbb{D}(0, 5/2) = \emptyset$$

for any $z \in J_{p_n} \setminus B(2, r)$ and for any $n \geq n_2$.

proof. Take $z \in J_{p_n} \setminus B(2, r)$ and set $(z_k, w_k) = f^k_n(z, w)$. Let $j \leq N-1$ be the minimum of k with $\operatorname{Re} z_k \leq 1$, which is assured by Lemma 3.11. Suppose $|\operatorname{Re} w_k| > 5/2$ for some $k \leq j$. Then $w_k \notin \mathbb{D}(0, 5/2)$, hence by Lemma 3.9, $w_N \notin \mathbb{D}(0, 5/2)$. Thus we may assume $|\operatorname{Re} w_k| \leq 5/2$ for $k \leq j$. We take δ and n_2 so that

$$64^N(\delta + \frac{4\epsilon_n}{63}) < \frac{\sqrt{6}}{10},$$

for any $n \geq n_2$.

Set $w = u + iv$ and $w_k = u_k + iv_k$. We show, by induction on k, $|v_k| < \dfrac{\sqrt{6}}{10}$ for $k \leq j$. Suppose it is true for $k = m$. Then it follows

$$\begin{aligned} |v_{m+1}| &= |4(u_m^2 - v_m^2)u_m v_m - 4\operatorname{Im} z_m| \\ &\leq 4(u_m^2 + v_m^2)|u_m v_m| + 4\epsilon_n \\ &\leq 4((\frac{5}{2})^2 + v_m^2)\frac{5}{2}|v_m| + 4\epsilon_n. \end{aligned}$$

By the induction hypothesis, we have $4((\dfrac{5}{2})^2 + v_m^2)\dfrac{5}{2} < \dfrac{631}{10} < 64$. Thus we

have $|v_{m+1}| < 64|v_m| + 4\epsilon_n$ and it follows

$$|v_{m+1}| < 64^{m+1}|v| + 4\epsilon_n \sum_{j=0}^{m} 64^j < 64^N(\delta + \frac{4\epsilon_n}{63}) < \frac{\sqrt{6}}{10}.$$

Thus the case $k = m+1$ holds.

Since $\mathrm{Re}\, z_j \le 1$ from the choice of j, we have

$$\begin{aligned} u_{j+1} &= (u_j^2 - v_j^2)^2 - 4u_j^2 v_j^2 + 4(2 - \mathrm{Re}\, z_j) \\ &\ge 4(2 - \mathrm{Re}\, z_j) - 4u_j^2 v_j^2 \\ &> 4 - 4 \cdot \frac{25}{4} \cdot \frac{6}{100} = \frac{5}{2}. \end{aligned}$$

Thus $w_{j+1} \notin \mathbb{D}(0, 5/2)$, hence $w_N \notin \mathbb{D}(0, 5/2)$ by Lemma 3.9, which proves the lemma. $\Box$

Set $S(0, 1/4, r) = \{x + yi \in \mathbb{C}; |x| \le 1/4, |y| \le r\}$.

Lemma 3.13. *Suppose $r < 7/128$. Then, for any $\delta' < 1/4$, there exists $n_3 \ge n_2$ so that for all $n \ge n_3$ and $z \in J_{p_n} \cap B(2, r)$, we have $q_z(S(0, 1/4, \delta')) \subset$ $int\, S(0, 1/4, \delta')$.*

proof. Fix $\delta' < 1/4$. Take n_3 so that $\epsilon_n \le \delta'/8$ for $n \ge n_3$. Take $w \in S(0, 1/4, \delta')$ and $z \in J_{p_n} \cap B(2, r)$. Then

$$\begin{aligned} |v_1| &\le 4|u^2 - v^2||uv| + 4\epsilon_n \\ &\le 4(\frac{1}{16} + \delta'^2)\frac{1}{4}\delta' + 4\epsilon_n < \delta'. \\ |u_1| &\le (u^2 - v^2)^2 + 4u^2v^2 + 4(2 - \mathrm{Re}\, z) \\ &\le (\frac{1}{4^2} + \delta'^2)^2 + \frac{4\delta'^2}{4^2} + 4r \\ &\le \frac{1}{8^2} + \frac{1}{4^3} + 4r < \frac{1}{4}. \end{aligned}$$

This completes the proof. $\Box$

Let δ and n_2 be given in Lemma 3.12 and let n_3 be given in Lemma 3.13.

Lemma 3.14. *For $n \ge n_3$, we have*
(a) $S(0, 1/4, \delta) \cap K_z = \emptyset$ for any $z \in J_{p_n} \setminus \{\beta_n\}$,
(b) $S(0, 1/4, \delta) \cap J_{\beta_n} = \emptyset$,
(c) $J_{p_n} \times S(0, 1/4, \delta) \subset (J_{p_n} \times \mathbb{C}) \setminus J_2$.

proof. (a) Any $z \in J_{p_n} \setminus \{\beta_n\}$ leaves $B(2,r)$ under finite iterations of p_n because β_n is a repelling fixed point of p_n. Set $m = min\{k; z_k := p_n^k(z) \notin B(2,r)\}$. By Lemma 3.13, it follows

$$Q_{n,z}^m(S(0,1/4,\delta)) \subset int\, S(0,1/4,\delta) \subset \{|\mathrm{Im}\, w| < \delta\}.$$

Then applying Lemma 3.12 to $z = z_m$, we have

$$Q_{n,z}^{m+N}(S(0,1/4,\delta)) \cap K_{z_{m+N}} = \emptyset.$$

By the invariance of K, we have $S(0,1/4,\delta) \cap K_z = \emptyset$.
(b) Since $p_n(\beta_n) = \beta_n$, Lemma 3.13 implies $q_{\beta_n}(S(0,1/4,\delta)) \subset int\, S(0,1/4,\delta)$. Then $S(0,1/4,\delta) \subset int\, K_{\beta_n}$ and we conclude $S(0,1/4,\delta) \cap J_{\beta_n} = \emptyset$.
(c) It follows from $(a), (b)$. $\square$

Lemma 3.15. *For $n \geq n_3$, $D_{J_{p_n}} \cap J_2 = \emptyset$.*

proof. Set $(z_k, w_k) = f_n^k(z, 0)$ for $z \in J_{p_n}$. If $z = \beta_n$, Lemma 3.13 says that $w_k \in S(0,1/4,\delta)$ for any $k \geq 0$ and by Lemma 3.14 (b), $\{(z_k, w_k)\}$ is uniformly bounded away from J_2.

For $z \in J_{p_n} \setminus \{\beta_n\}$, set m as above. If $m > 0$, Lemma 3.13 says $w_k \in int\, S(0,1/4,\delta)$ for $k \leq m$ and $\{(z_k, w_k); k \leq m\}$ is uniformly bounded away from J_2. Applying Lemma 3.12 to $z = z_m$, $\{(z_k, w_k); k \geq m+N\}$ is uniformly bounded away from J_2. The fact that $\{(z_k, w_k); m < k < m+N\}$ is uniformly bounded away from J_2 follows from the compactness of J_{p_n} and the uniformity of N. If $m = 0$, i.e., $z \notin B(2,r)$, applying Lemma 3.12 to z, we get the same conclusion. Thus we have shown that $D_{J_{p_n}} \cap J_2 = \emptyset$. $\square$

Let δ and n_2 be given in Lemma 3.12 and let n_3 be given in Lemma 3.13. We may assume $\delta < 1/4$ and $n_3 \geq n_2$.

Theorem 3.5. *Set $f_n(z,w) = (p_n(z), q(z,w))$ as above. For $n \geq n_3$, f_n is Axiom A and satisfies*
(a) $C_1 = \{(\beta_n, 0)\}$, $\quad C_0 = C_{J_{p_n}} \setminus C_1$.
(b) $\Lambda = \Lambda_1 = \{(\beta_n, \alpha_n)\}$ is itself a basic set, where α_n is the attracting fixed point of q_{β_n}.
(c) J_z is disconnected for all $z \in J_{p_n} \setminus \{\beta_n\}$ and is a quasicircle for $z = \beta_n$.
(d) $A_{pt}(C_{J_{p_n}}) = A_{cc}(C_{J_{p_n}}) \neq A(C_{J_{p_n}})$.

proof. By Theorem 2.1 and Lemmas 3.10 and 3.15, f_n for $n \geq n_3$ are Axiom A.
(a) Lemma 3.14, (a) says that $K \cap (C_{J_{p_n}} \setminus \{(\beta_n, 0)\}) = \emptyset$. By Lemma 3.13, it follows that $(\beta_n, 0) \in K$. These imply (a).

(b) By Lemma 3.13, q_{β_n} has an attracting fixed point. From (a), there is no saddle periodic point for $z \neq \beta_n$.
(c) The case $z \in J_{p_n} \setminus \{\beta_n\}$ follows from (a) and Proposition 2.3 in [J3]. If $z = \beta_n$, $q_z(w) = w^4 + 4(2-\beta_n)$ has an attracting fixed point α_n. Thus J_{β_n} is a quasicircle as a quasiconformal image of the Julia set of $w \mapsto w^4$.
(d) Since $C_{J_{p_n}} = J_{p_n} \times \{0\}$, any $C \in \mathcal{C}(C_{J_{p_n}})$ is of the form $J \times \{0\}$, where J is a connected component of J_{p_n}. Recall that the component of J_{p_n} containing β_n is just a single point $\{\beta_n\}$ itself. Thus (a) implies $C \subset C_0$ or $C \subset C_1$. Now by Theorem 1.1, we conclude $A_{cc}(C_{J_{p_n}}) = A_{pt}(C_{J_{p_n}})$.

Since C_0 is not closed in C_{J_p}, it follows from Theorem 1.2, that $A(C_{J_{p_n}}) \neq A_{pt}(C_{J_{p_n}})$, Thus we conclude $A_{pt}(C_{J_{p_n}}) = A_{cc}(C_{J_{p_n}}) \neq A(C_{J_{p_n}})$. $\square$

References

[DH1] L. DeMarco & S. Hruska: Axiom A polynomial skew products of $\mathbb{C}^2$ and their postcritical sets. Ergod. Th. & Dynam. Sys. 28 (2008), pp. 1749–1779.

[DH2] L. DeMarco & S. Hruska: Axiom A polynomial skew products of $\mathbb{C}^2$ and their postcritical sets - Erratum. Ergod. Th. & Dynam. Sys. 31 (2011), pp. 631–636.

[FS1] J.E. Fornaess & N. Sibony: Complex Dynamics in higher dimension II. Ann. Math. Studies 137 (1995), pp. 134–182.

[FS2] J.E. Fornaess & N. Sibony: Hyperbolic maps on $\mathbb{P}^2$. Math. Ann. 311 (1998), pp. 305–333.

[J1] M. Jonsson: Dynamical studies in several complex variables, I. Hyperbolic dynamics of endomorphisms. PhD thesis, Royal Institute of Technology, 1997.

[J2] M. Jonsson: Holomorphic motions of hyperbolic sets. Michigan Math. J. 45 (1998), pp. 409–415.

[J3] M. Jonsson: Dynamics of polynomial skew products on $\mathbb{C}^2$. Math. Ann. 314 (1999), pp. 403–447.

[Mih] E. Mihailescu: The set K^- for hyperbolic non-invertible maps. Erg. Th. & Dyn. Sys. 22 (2002), pp. 873–887.

[MU] E. Mihailescu & M. Urbański: Estimates for the stable dimension for holomorphic maps. Houston J. Math. 31 (2005), pp. 367–389.

[Mil] J. Milnor: *Dynamics in One Complex Variable.* Third Ed. Annals of Math. Studies 160. Princeton Univ. Press (2006).

[P] F. Przytycki: Anosov endomorphisms. Studia Math. 58 (1976), pp. 249–285.

[QY] W. Qiu & Y. Yin: Proof of the Branner-Hubbard conjecture on Cantor Julia sets. Science in China, Series A, 52 (2009), pp. 45–65.

[R] D. Ruelle: *Elements of Differentiable Dynamics and Bifurcation Theory.* Academic Press, New York, (1989).

[Se] O. Sester: Hyperbolicitè des polynômes fibrés. Bull. Soc. Math. France 127 (1999), pp. 393–428.

[Su] H. Sumi: Dynamics of postcritically bounded polynomial semigroups III : Classification of hyperbolic semigroups and random Julia sets which are Jordan curves but not quasicircles. Ergod. Th. & Dynam. Sys. 30 (2010), pp. 1869–1902.